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The statistical characteristics of the displacement front in a randomly heterogeneous medium☆

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Abstract

The problem of the displacement of oil by water in a randomly hetrogeneous medium with specified statistical characteristics is considered. Using an improved perturbation theory within the framework of the Buckley – Leverret two-phase model, a statistical solution is constructed which enables the average shape of the displacement front to be obtained, and also the variance of the saturation and the longitudinal velocities on the displacement front, together with a variogram of the fluctuations of the front shape.

The mathematical description of the displacement of oil by water[1] is based on the theory of two-phase seepage of immiscible liquids in a porous medium. When considering a randomly heterogeneous medium, when the conductivity (and also the porosity) is a certain random function of the coordinates, the water saturation, the seepage rate and other characteristics of the displacement are also random functions of the coordinates. It is of interest to find their average values, the variances and statistical moments of different orders, which are essential, for example, for estimating the probability of water breaking through into the oilwell.

When using one of the most widely employed approaches to solving this problem, namely, the Monte Carlo method, a deterministic problem for the specified form of the conductivity field is first solved numerically, and then averaging is carried out over the samples using the statistical properties of the conductivity field, assumed to be given. To obtain sufficient accuracy when obtaining these statistical characteristics, detailed numerical calculations are necessary for each sample of the conductivity field and the use of a large number of samples for further averaging, which leads to considerable computer costs and limits the possibilities of simulation.

An analytical approach to the problem consists of finding statistical solutions of stochastic differential equations and reduces to obtaining a system of equations for the statistical moments and their subsequent solution. The problem then arises of terminating the infinite chain of equations for the statistical moments using some closure hypothesis.

As it applies to single-phase flows, the closure procedure is carried out by introducing effective characteristics of the medium, which enable the behaviour of the system to be described over a large-scale medium taking into account the average effect of very small-scale processes. In the majority of cases, the effective conductivity is calculated using the lowest approximation of perturbation theory; it can be improved by summing a certain infinite subsequence of

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a perturbation-theory series ("the improved perturbation theory"[2–4]) or using the renormalization-group method,[5–7] known in the theory of random media as "upscaling".

As it applies to two-phase flows, the pattern is complicated by the occurrence on the displacement front of inhomogeneous finger-shaped structures, related to Saffman - Taylor instability.[8] A mathematical analysis of the problem is difficult due to the fact that the conductivity of the medium depends on the water saturation, and this leads to a non-linear relation between the water saturation and the seepage rate (called the "viscous coupling").[9] However, in the majority of paper on two-phase seepage flows the simplified problem of the evolution of water saturation in a given field of random velocities, the statistical characteristics of which were assumed to be known, was solved, or the random-velocity field was determined from the solution of the problem of single-phase flow in a randomly heterogeneous medium.[10–12] A number of authors[13,14] have drawn attention to the importance of taking the viscous coupling into account. Viscous coupling was considered in the lowest approximation of perturbation theory in Ref.15, and was also considered within the framework of the improved perturbation theory in Ref.16. In the linear approximation, an equation was obtained in Ref.16 for the seepage rate, the solution of which enabled the dispersion of the longitudinal shifts of the displacement front and the average shape of the front to be determined.

Using the formalism developed earlier in Ref.16, below we calculate the variance of the longitudinal velocities and the variance of the water saturation on the displacement front, and also variograms of the longitudinal displacements of the front surface.

1. Formulation of the problem

The Buckley – Leverett system of equations,[1,17] describing the process of displacement of oil by water, is the basis of the mathematical model of the process of two-phase seepage, neglecting gravitational and capillary phenomena. This system consists of the water-saturation balance, the law of conservation of mass (volume) of both phases and the generalized Darcy law for a two-phase system

$$\phi\frac{\partial S}{\partial t} + F'(S)\mathbf{u}\nabla S = 0, \quad \nabla\mathbf{u} = 0, \quad \mathbf{u} = -\lambda\nabla p \tag{1.1}$$

Here ϕ is the porosity, S is the water saturation (the fraction of the porous space filled with water), $u = u_1 + u_2$ is the total seepage rate of both phases (water and oil), $F(S)$ is the Buckley – Leverett flow distribution function, which depends only on the water saturation (the form of this function is determined by the ratio of the mobilities of both phases[1]), p is the pressure and λ is the generalized conductivity of the two-phase system, which depends on the mobilities and the relative permeabilties of each of the phases.[17]

It is assumed that the generalized conductivity λ depends on the saturation S, and in the case of a randomly heterogeneous medium is also a random function of the coordinates. It is usually assumed that

$$\lambda(\mathbf{r}, S) = \kappa(\mathbf{r})K(S) \tag{1.2}$$

where $\kappa(r)$ is a random function of the coordinates with specified statistical properties. Henceforth we will assume that $\phi = \text{const}$, which corresponds to replacing the porosity function by its average value and does not essentially reduce the generality of the analysis.

It should be noted that, in view of the dependence of the conductivity λ on the saturation, according to formula (1.2), the seepage rate u depends on S, as a result of which Eq. (1.1) are strongly related, and a consideration of this relation ("viscous coupling") is extremely important.

In a homogeneous medium the solution of the problem of the motion of the displacement front is constructed by the method of characteristics, where it turns out that the solution for water saturation must be sought in the class of generalized solutions containing discontinuities on the displacement front.[1] Henceforth we will confine ourselves to investigating the behaviour of the discontinuity surface (the shape of the displacement front) in a randomly inhomogeneous medium on the assumption that on both sides of the discontinuity surface the values of the water saturation are specified constants S_1 and S_2. The problem will be considered in a space of arbitrary dimensions d, but in the one-dimensional case the solution is trivial, and we will consider the two-dimensional problem ($d = 2$) and three-dimensional problem ($d = 3$) as special cases. We will introduce the following notation below. The set of spatial coordinates is a d-dimensional vector $\mathbf{r}$ with components x_i (the Latin subscripts take values $i = 1, \ldots, d$), the x_1 axis is directed along the average direction of the front propagation, and the components of the $(d-1)$-dimensional vector $\mathbf{y}$, orthogonal to the

x_1 axis, will be denoted by x_α, where the Greek subscripts take values ($\alpha = 2, \ldots, d$). In this notation, the gradient vector has components $\partial_i = \{\partial_1, \partial_\alpha\}$, the wave vector $\mathbf{Q} = \{q_1, \mathbf{q}\}$, which arises when carrying out Fourier transformations, has components q_1 and q_α.

We will write the equation of the interface in the form

$$\varphi(x_1, \mathbf{y}, t) = x_1 - h(\mathbf{y}, t) = 0 \tag{1.3}$$

The region occupied by the first liquid (water) corresponds to the value $\varphi < 0$, while $\varphi > 0$ corresponds to the region occupied by the second liquid (oil); the front and rear sides of the discontinuity surface correspond to $\varphi = +0$ and $\varphi = -0$.

From the first equation of (1.1) at the interface we can obtain an equation for the function $h(\mathbf{y}, t)$, defining the shape of this surface[16]

$$\partial_t h(\mathbf{y}, t) = \frac{c_0}{u_0}(u_1 - u_\alpha \partial_\alpha h)\big|_{\varphi = 0}, \quad c_0 = \frac{u_0}{\phi}\frac{F(S_1) - F(S_2)}{S_1 - S_2} \tag{1.4}$$

where u_0 is the mean seepage rate directed along the x_1 axis.

In the linear approximation, the equation for the function $\delta h = h - c_0 t$ can be written in the form

$$\partial_t \delta h = L\delta h + f \tag{1.5}$$

where L is a certain linear integro-differential operator;[16] it is constructed within the framework of the "improved perturbation theory", when the value of the fluctuations of the logarithm of the conductivity $\ln[\kappa(\mathbf{r})/\kappa 0]$ plays the role of the parameter of the expansion in series of perturbation theory, rather-than the conductivity fluctuations $\delta\kappa(\mathbf{r})/\langle\kappa(c)\rangle$.[7,18,19] The function δh describes the perturbations of the plane surface of the displacement front, while f is a random function which depends on $\kappa(\mathbf{r})$.

In Fourier-transform space the solution of Eq. (1.5) is represented in the form[16]

$$\delta h(\mathbf{q}, \omega) = M(q_1, \mathbf{q})\frac{\mu(q_1, \mathbf{q})}{c_0}, \quad q_1 = -\frac{\omega}{c_0}, \quad \mu(\mathbf{r}) = \ln\left[\frac{\kappa(\mathbf{r})}{\kappa_0}\right] \tag{1.6}$$

$$M(q_1, \mathbf{q}) = \frac{q^2}{(\imath q_1 + Aq)(q_1^2 + q^2)}\left(1 + \frac{\imath q_1 A}{q}\right), \quad A = \frac{K^2(S_1) - K^2(S_2)}{4K(S_1)K(S_2)} \tag{1.7}$$

In the expression for $\mu(\mathbf{r})$ the constant κ_0 is chosen in such a way that $\langle\mu(\mathbf{r})\rangle = 0$ (here and henceforth the angle brackets denote the average over the ensemble of samples of the conductivity field); in this case, κ_0 is the geometric mean of the random function $\kappa(\mathbf{r})$ at the point $\mathbf{r}$ (by virtue of the statistical homogeneity of the conductivity field this quantity is independent of $\mathbf{r}$).

We can calculate the dispersion of the shift in the displacement front from the formulae[16]

$$B_0 = \langle\delta h^2(\mathbf{r})\rangle = \frac{1}{2\sqrt{\pi}}\frac{\Gamma(d/2 - 1)}{\Gamma((d-1)/2)}I(d, A)\int_0^\infty D(r)\left[1 - \left(\frac{r}{r_0}\right)^{d-2}\right]rdr \tag{1.8}$$

$$I(d, A) = \int_0^\pi \Theta(\theta)\sin^d\theta d\theta, \quad \Theta(\theta, A) = \frac{\sin^2\theta + A^2\cos^2\theta}{\cos^2\theta + A^2\sin^2\theta} \tag{1.9}$$

Here $D(r) = \langle\mu(\mathbf{r})\mu(0)\rangle$ is the paired correlation function of the log-conductivity. The arbitrary parameter r_0 is related to the non-uniqueness of the solution of the equation for Green's function of the Laplace operator. This problem was discussed in detail in Ref.16, formulae for $I(2, A)$ and $I(3, A)$ were derived, it was shown that the choice of the form of the correlation function $D(r)$ has no appreciable effect on the result for B_0, and it was also pointed out that the case $d = 2$ is a singular case in view of the logarithmic increase in Green's function of the two-dimensional Laplace equation.

2. The variance of the longitudinal velocities on the displacement front

According to solution (1.6), the Fourier transform of the longitudinal velocities on the displacement front has the form

$$\upsilon_1(\mathbf{q},\omega) = -\iota\omega\delta h(\mathbf{q},\omega) = -iq_1 M(q_1,\mathbf{q})\mu(q_1,\mathbf{q}) \quad (2.1)$$

Hence we can obtain the variance of the velocities on the front surface

$$T_0 = \langle\upsilon_1^2(\mathbf{y},t)\rangle = \int\langle\upsilon_1(\mathbf{q},\omega)\upsilon_1(\mathbf{q}',\omega')\rangle\exp\{\iota(\mathbf{q}+\mathbf{q}')\mathbf{y}-\iota(\omega+\omega')t\}\frac{d\mathbf{q}d\omega}{(2\pi)^d}\frac{d\mathbf{q}'d\omega'}{(2\pi)^d} =$$

$$= -c_0^2\int q_1 q_1' M(q_1,\mathbf{q})M(q_1',\mathbf{q}')\langle\mu(q_1,\mathbf{q})\mu(q_1',\mathbf{q}')\rangle\times \quad (2.2)$$

$$\times\exp\{\iota(\mathbf{q}+\mathbf{q}')\mathbf{y}-\iota c_0(q_1+q_1')t\}\frac{d\mathbf{q}dq_1}{(2\pi)^d}\frac{d\mathbf{q}'dq_1'}{(2\pi)^d}$$

Using the formula

$$\langle\mu(q_1,\mathbf{q})\mu(q_1',\mathbf{q}')\rangle = (2\pi)^d\delta(q_1+q_1')\delta(\mathbf{q}+\mathbf{q}')D(q_1,\mathbf{q})\equiv$$
$$\equiv(2\pi)^d\delta(\mathbf{Q}+\mathbf{Q}')D(\mathbf{Q}) \quad (2.3)$$

and the property $M(-q_1,-\mathbf{q}) = M^*(q_1,\mathbf{q})$, which follows from the first equation of (1.7), we obtain

$$T_0 = c_0^2\int q_1^2|M(q_1,\mathbf{q})|^2 D(q_1,\mathbf{q})\frac{d\mathbf{q}dq_1}{(2\pi)^d} \quad (2.4)$$

where $D(\mathbf{Q})$ is a specified function, which, in an isotropic medium, depends on $q_1^2+q^2 = Q^2$, which enables us to represent it in the form of a Laplace integral

$$D(\mathbf{Q}) = \int_0^\infty\rho(m)\exp\{-mQ^2\}dm \quad (2.5)$$

The function $\rho(m)$ defines the form of the correlation function of the log-conductivity $D(\mathbf{r})$.

Using the notation $q_1 = Q\cos\theta$, $q = Q\sin\theta$ and formulae (1.7), (2.3) and (2.5), we obtain

$$T_0 = c_0^2\frac{\Omega_{d-1}}{(2\pi)^d}I_1(d,A)\int_0^\infty\rho(m)dm\int_0^\infty\exp\{-mQ^2\}Q^{d-1}dQ \quad (2.6)$$

$$I_1(d,A) = \int_0^\pi\Theta(\theta)\sin^d\theta\cos^2\theta d\theta = I(d,A)-I(d+2,A) =$$
$$= \frac{d+1+A}{1-A^2}B\left(\frac{d+1}{2},\frac{1}{2}\right)-\frac{A^2}{1-A^2}I(d,A),\quad \Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \quad (2.7)$$

Here $B(a, b)$ is Euler's beta-function and Ω_d is the area of the surface of the d-dimensional sphere of unit radius. The integral over Q is expressed in terms of the gamma-function, while the integral $I(d, A)$, defined by relation (1.9), in general is expressed in terms of hypergeometric functions, but in the special cases when $d = 2, 3$ is expressed in terms of elementary functions.[16]

As was done previously (see the Appendix in Ref.16), it can be shown that

$$\int_0^\infty\frac{\rho(m)dm}{(4\pi m)^{d/2}} = D(\mathbf{r})|_{r=0} = D_0 \quad (2.8)$$

where D_0 is the variance of the log-conductivity of the medium.

Using relations (2.6) and (2.8) after the calculations leads to the following expression for the variance of the velocities

$$T_0 = \frac{c_0^2 D_0 I_1(d, A)}{B((d-1)/2, 1/2)} \tag{2.9}$$

3. Variance of the water saturation in the region of the displacement front

Assuming that the water saturation distribution near the displacement front has the form of a discontinuous function, which experiences a jump at the displacement front, the expression for $S(\mathbf{r}, t)$ can be written in the form of the generalized function

$$S(\mathbf{r}) = \frac{S_1 + S_2}{2} - \frac{S_1 - S_2}{2}\operatorname{sign}[x_1 - c_0 t - \delta h(\mathbf{y}, t)] \tag{3.1}$$

To calculate the average distribution of the water saturation we must average expression (3.1) over the fluctuations in the shape of the interface δh, i.e. calculate the value of $\langle \operatorname{sign}[x_1 - c_0 t - \delta h(\mathbf{y}, t)]\rangle$. To do this it is convenient first of all to obtain its derivative with respect to x_1. Using the formula $\partial_x \operatorname{sign} x = 2\delta(x)$ and representing the δ-function in the form of a Fourier integral, we obtain

$$\begin{aligned}\langle \delta[x_1 - c_0 t - \delta h(\mathbf{y}, t)]\rangle &= \int \exp\{\iota p(x_1 - c_0 t)\}\langle \exp\{-\iota p \delta h(\mathbf{y}, t)\}\rangle \frac{dp}{2\pi} \equiv \\ &\equiv \int \exp\{ip(x_1 - c_0 t)\}\Psi[g(p, \mathbf{z}, \tau)]\frac{dp}{2\pi}, \quad g(p, \mathbf{z}, \tau) = -p\delta(\mathbf{z} - \mathbf{y})\delta(\tau - t)\end{aligned} \tag{3.2}$$

$$\Psi[g(p, \mathbf{z}, \tau)] = \langle \exp\{\iota \int \delta h(\mathbf{z}, \tau) g(p, \mathbf{z}, \tau) d\mathbf{z} d\tau\}\rangle \tag{3.3}$$

where $\Psi[g(p, \mathbf{z}, \tau)]$ is the characteristic functional of the random field $\delta h(\mathbf{y}, t)$.

The form of the characteristic functional for a normal distribution of the random field has the form

$$\begin{aligned}\Psi[g(p, \mathbf{z}, \tau)] &= \exp\left\{-\frac{1}{2}\int g(p, \mathbf{z}, \tau) B(\mathbf{z}, \tau; \mathbf{z}', \tau') g(p, \mathbf{z}', \tau') d\mathbf{z} d\tau d\mathbf{z}' d\tau'\right\} \\ B(\mathbf{z}, \tau; \mathbf{z}', \tau') &= \langle \delta h(\mathbf{z}, \tau)\delta h(\mathbf{z}', \tau)\rangle\end{aligned} \tag{3.4}$$

which, taking into account the form of the function $g(p, \mathbf{z}, \tau)$, gives

$$\Psi[g(p, \mathbf{z}, \tau)] = \exp\left\{-\frac{1}{2}B(\mathbf{y}, t; \mathbf{y}, t)p^2\right\}, \quad B(\mathbf{y}, t; \mathbf{y}, t) = \langle \delta h^2(\mathbf{y}, t)\rangle = B_0 \tag{3.5}$$

where B_0 is the variance of the longitudinal shifts of the front.

After integrating over p and then over x_1, we obtain

$$\langle \operatorname{sign}[x_1 - c_0 t - \delta h(\mathbf{y}, t)]\rangle = \operatorname{erf}\left(\frac{x_1 - c_0 t}{\sqrt{2B_0}}\right) \tag{3.6}$$

which, after averaging (3.1) and taking (3.6) into account, leads to a formula for the mean distribution of the water saturation.[16]

Using representation (3.1) and the obvious relation $\operatorname{sign}^2(x) = 1$, we obtain for the mean square of the water saturation

$$\langle S^2(\mathbf{r}, t)\rangle = \frac{S_1^2 + S_2^2}{2} - \frac{S_1^2 - S_2^2}{2}\operatorname{erf}\left(\frac{x_1 - c_0 t}{\sqrt{2B_0}}\right) \tag{3.7}$$

which leads to the required formula for the variance of the water saturation

$$\sigma_S^2 = \langle S^2(\mathbf{r}, t)\rangle - \langle S(\mathbf{r}, t)\rangle^2 = \left(\frac{(S_1 - S_2)}{2}\right)^2\left\{1 - \operatorname{erf}^2\left(\frac{x_1 - c_0 t}{\sqrt{2B_0}}\right)\right\}$$

4. Variogram of the longitudinal shifts of the displacement front

One of the characteristics of the random field $f(\mathbf{r}, t)$ widely used in geostatics is the variogram, which is defined by the relation

$$\gamma(\mathbf{r}, t; \mathbf{r}', t') = \langle [f(\mathbf{r}, t) - f(\mathbf{r}', t')]^2 \rangle$$

and for a stationary and statistically homogeneous field, when

$$\langle f(\mathbf{r}, t) f(\mathbf{r}', t') \rangle = B(\mathbf{r} - \mathbf{r}', t - t')$$

the variogram is simply related to the paired correlation function

$$\gamma(\mathbf{r}, t; \mathbf{r}', t') = 2[B_0 - B(\mathbf{r} - \mathbf{r}', t - t')]$$

where $B_0 = B(0, 0)$ is the variance of the random field $f(\mathbf{r}, t)$.

We will consider the variogram for δh – perturbations of the shape of the plane displacement front $x_1 - c_0 t = 0$

$$\gamma(\mathbf{y}) = \langle [\delta h(\mathbf{y}) - \delta h(0)]^2 \rangle = 2[B_0 - B(\mathbf{y})] \tag{4.1}$$

In the two-dimensional case, the integral expressions for B_0 and $B(y)$ contain logarithmic divergences, which lead to non-uniqueness (the dependence of these quantities on the normalization point r_0), which are discussed in detail in Ref.16, but this problem does not arise for variograms.

We will calculate the paired correlation function

$$B(\mathbf{y}) = \langle \delta h(\mathbf{y}, t) \delta h(0, t) \rangle = \int |M(q_1, \mathbf{q})|^2 D(q_1, \mathbf{q}) \exp\{ \imath \mathbf{q}\mathbf{y} \} \frac{dq_1 d\mathbf{q}}{(2\pi)^d}$$

Using formulae (1.7) and (2.5) for the Fourier transforms $\delta h(\mathbf{y}, \omega)$ and $D(q_1, \mathbf{q})$, we obtain

$$\begin{aligned} B(\mathbf{y}) &= \int_0^\infty \rho(m) dm \int \left\{ \frac{1 + A^2}{1 - A^2} \left[\frac{1}{q_1^2 + A^2 q^2} - \frac{1}{q_1^2 + q^2} \right] - \frac{q^2}{(q_1^2 + q^2)^2} \right\} \times \\ &\times \exp\{ -m(q_1^2 + q^2) + \imath \mathbf{q}\mathbf{y} \} \frac{dq_1 d\mathbf{q}}{(2\pi)^d} \end{aligned} \tag{4.2}$$

To carry out the integrations over q_1 and $\mathbf{q}$ we use the formulae

$$\frac{1}{(q^2)^n} = \frac{1}{\Gamma(n)} \int_0^\infty \exp\{ -\eta q^2 \} \eta^{n-1} d\eta$$

$$J(q) = \int_{-\infty}^{\infty} \exp\{ -\eta q^2 + \imath q y \} \frac{dq}{2\pi} = \frac{1}{\sqrt{4\pi\eta}} \exp\left\{ -\frac{y^2}{4\eta} \right\}$$

$$\int_{-\infty}^{\infty} \exp\{ -\eta q^2 + \imath q y \} \frac{q^2 dq}{2\pi} = -\frac{d^2 J(y)}{dy^2}$$

After carrying out the integrations over the components of the vector q, substituting $\beta = 1/(1 + \eta)$ and integrating by parts, we obtain the function $B(\mathbf{y})$, and from formula (4.1) for the variogram we obtain

$$\begin{aligned} \gamma(\mathbf{y}) &= 2 \int_0^\infty \frac{m\rho(m) dm}{(4\pi m)^{d/2}} \int_0^1 \chi(\beta, A) \left[1 - \exp\left(-\frac{y^2}{4m} \beta \right) \right] \beta^{d/2 - 2} d\beta \\ \chi(\beta, A) &= \frac{1 + A^2}{1 - A^2} \left[\frac{1}{A\sqrt{1 - (1 - A^2)\beta}} - 1 \right] - \frac{1}{2}(1 + \beta) \end{aligned} \tag{4.3}$$

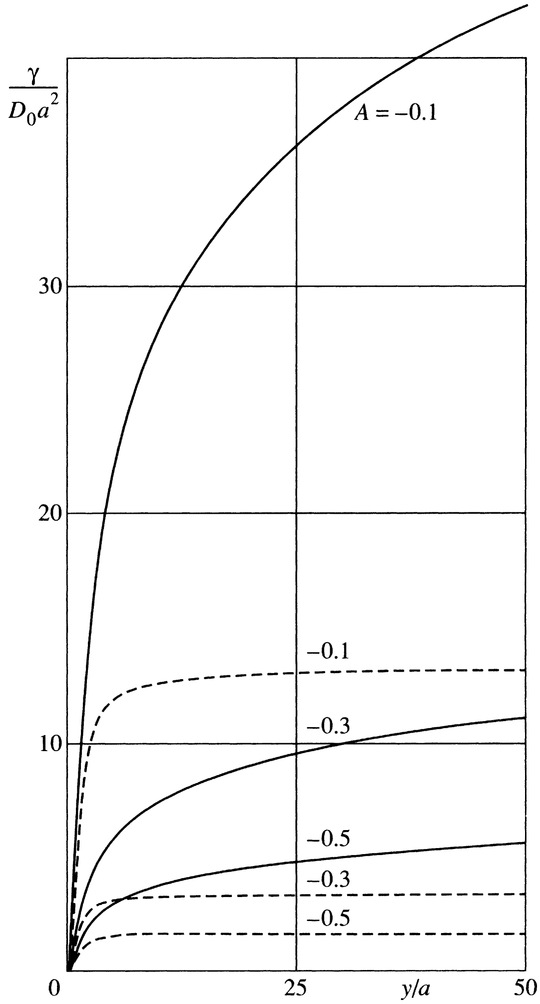

Fig. 1.

In the case when the paired correlation function of the log-conductivity has the form $D(\mathbf{r}) = D_0 \exp(-r^2/a^2)$, the function $\rho(m)$ is given by the relation $\rho(m) = D_0(4\pi m)^{d/2}\,\delta(m - a^2/4)$ and formula (4.3) simplifies to

$$\gamma(\mathbf{y}) = \frac{D_0 a^2}{2}\int_0^1 \chi(\beta, A)\left[1 - \exp\left(-\frac{y^2}{a^2}\beta\right)\right]\beta^{d/2-2}d\beta \tag{4.4}$$

Note that, according to the general principle of the weakening of correlations at considerable distances, we have the limit relations $B(\mathbf{y}) \to 0$ and $\gamma(\mathbf{y}) \to 2B_0$ when $y \to \infty$.

The results of a calculation of the variograms in the three-dimensional problem for three different values of the parameter A are shown by the dashed curves in the Fig. 1.

The case $d = 2$ is a special case in view of the characteristics of the behaviour of Green's function for Laplace equation (the logarithmic increase at long distances). Corresponding to this for large y the variogram also increases logarithmically. To estimate the asymptotic form in this case, we bear in mind that in the two-dimensional problem the neighbourhood of the point $\beta = 0$ makes the main contribution to the integral over β. Hence, for large y^2/a^2 the inverse value of the radical in the expression for $\chi(\beta)$ can be taken in the form

$$[1 - (1 - A^2)\beta]^{-1/2} \approx 1 + (1 - A^2)\beta/2$$

As a result, after reduction we obtain

$$\gamma(y) = \frac{D_0 a^2}{4}\int_0^{(y/a)^2}\left[\frac{A^2-A+2}{A+1}\frac{1-\exp(-t)}{t}+(A^2-A+1)\left(\frac{a}{y}\right)^2(1-\exp(-t))\right]dt = \\ = \frac{D_0 a^2}{2}\frac{A^2-A+2}{A(A+1)}\ln\frac{y}{a}+O(1) \qquad (4.5)$$

i.e. a function that increases logarithmically for large y/a. When obtaining formula (4.5) we took into account the relation

$$\int_0^x \frac{1-\exp(-t)}{t}dt = C_E + \ln x + \int_x^\infty \frac{\exp(-t)}{t}dt \text{ is Euler's constant}$$

The results of the calculation of the variograms using formula (4.4) for $d=2$ are shown in Fig. 1 by the continuous curves. It can be seen that in this case, for long distances, the variograms increase logarithmically in agreement with the asymptotic formula (4.5).

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